

# **Lecture -3**

## **Fourier Transform Properties**

# Fourier Transform Properties and Examples

## Objectives:

1. Properties of a Fourier transform
  - **Linearity & time shifts**
  - **Differentiation**
  - **Convolution** in the frequency domain

While the **Fourier series/transform** is very important for representing a signal in the **frequency domain**, it is also important for **calculating a system's response** (convolution).

- A **system's transfer function** is the **Fourier transform** of its **impulse response**
- Fourier transform of a signal's **derivative** is **multiplication** in the **frequency domain**:  $j\omega X(j\omega)$
- Convolution in the time domain is given by **multiplication** in the **frequency domain** (similar idea to log transformations)

# Review: Fourier Transform

A CT signal  $x(t)$  and its frequency domain, Fourier transform signal,  $X(j\omega)$ , are related by

$$X(j\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t} dt \quad \text{analysis}$$

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(j\omega)e^{j\omega t} d\omega \quad \text{synthesis}$$

This is denoted by:

$$x(t) \overset{F}{\leftrightarrow} X(j\omega)$$

For example:

$$e^{-at}u(t) \overset{F}{\leftrightarrow} \frac{1}{a + j\omega}$$

Often you have tables for common Fourier transforms

The Fourier transform,  $X(j\omega)$ , represents the **frequency content** of  $x(t)$ .

It exists either when  $x(t) \rightarrow 0$  as  $|t| \rightarrow \infty$  or when  $x(t)$  is periodic (it generalizes the Fourier series)

# Linearity of the Fourier Transform

The Fourier transform is a **linear function** of  $x(t)$

$$x_1(t) \stackrel{F}{\leftrightarrow} X_1(j\omega)$$

$$x_2(t) \stackrel{F}{\leftrightarrow} X_2(j\omega)$$

$$ax_1(t) + bx_2(t) \stackrel{F}{\leftrightarrow} aX_1(j\omega) + bX_2(j\omega)$$

This follows directly from the definition of the Fourier transform (as the integral operator is linear) & it easily extends to an arbitrary number of signals

Like impulses/convolution, if we know the Fourier transform of simple signals, we can calculate the Fourier transform of more complex signals which are a linear combination of the simple signals

# Fourier Transform of a Time Shifted Signal

We'll show that a Fourier transform of a signal which has a **simple time shift** is:

$$F\{x(t-t_0)\} = e^{-j\omega t_0} X(j\omega)$$

i.e. the original Fourier transform but **shifted in phase** by  $-\omega t_0$

## Proof

Consider the Fourier transform synthesis equation:

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(j\omega) e^{j\omega t} d\omega$$

$$\begin{aligned} x(t-t_0) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} X(j\omega) e^{j\omega(t-t_0)} d\omega \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \left( e^{-j\omega t_0} X(j\omega) \right) e^{j\omega t} d\omega \end{aligned}$$

but this is the synthesis equation for the Fourier transform

$$e^{-j\omega_0 t} X(j\omega)$$

# Example: Linearity & Time Shift

Consider the signal (linear sum of two time shifted rectangular pulses)

$$x(t) = 0.5x_1(t - 2.5) + x_2(t - 2.5)$$

where  $x_1(t)$  is of width 1,  $x_2(t)$  is of width 3, centred on zero (see figures)

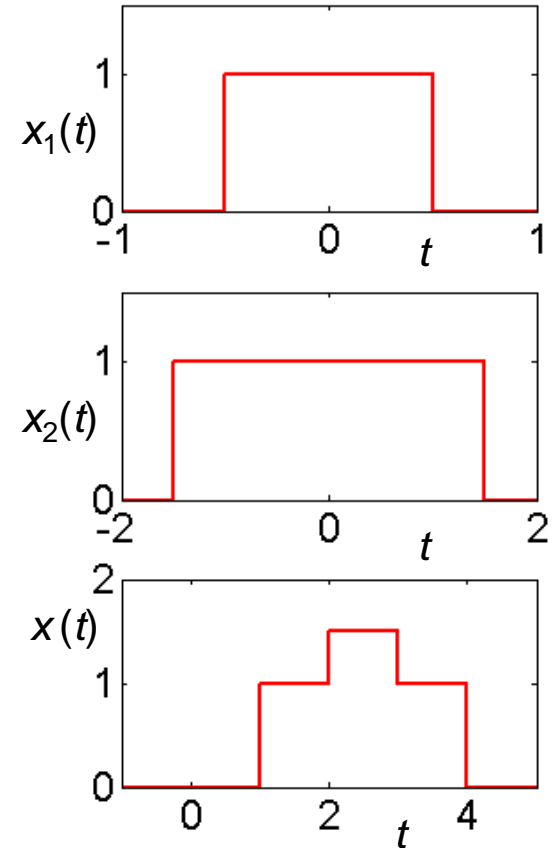
Using the FT of a rectangular pulse L10S7

$$X_1(j\omega) = \frac{2 \sin(\omega/2)}{\omega}$$

$$X_2(j\omega) = \frac{2 \sin(3\omega/2)}{\omega}$$

Then using the **linearity** and **time shift** Fourier transform properties

$$X(j\omega) = e^{-j5\omega/2} \left( \frac{\sin(\omega/2) + 2 \sin(3\omega/2)}{\omega} \right)$$



# Fourier Transform of a Derivative

By differentiating both sides of the Fourier transform synthesis equation with respect to  $t$ :

$$\frac{dx(t)}{dt} = \frac{1}{2\pi} \int_{-\infty}^{\infty} j\omega X(j\omega) e^{j\omega t} d\omega$$

Therefore noting that this is the synthesis equation for the Fourier transform  $j\omega X(j\omega)$

$$\frac{dx(t)}{dt} \stackrel{F}{\leftrightarrow} j\omega X(j\omega)$$

This is very important, because it replaces **differentiation** in the **time domain** with **multiplication** (by  $j\omega$ ) in the **frequency domain**.

We can **solve ODEs** in the **frequency domain** using **algebraic** operations (see next slides)



# Convolution in the Frequency Domain

We can easily solve ODEs in the frequency domain:

$$y(t) = h(t) * x(t) \xleftrightarrow{F} Y(j\omega) = H(j\omega)X(j\omega)$$

Therefore, to apply **convolution in the frequency domain**, we just have to **multiply the two Fourier Transforms**.

To solve for the differential/convolution equation using Fourier transforms:

1. Calculate **Fourier transforms** of  $x(t)$  and  $h(t)$ :  $X(j\omega)$  by  $H(j\omega)$
2. **Multiply**  $H(j\omega)$  by  $X(j\omega)$  to obtain  $Y(j\omega)$
3. Calculate the **inverse Fourier transform** of  $Y(j\omega)$

$H(j\omega)$  is the LTI system's **transfer function** which is the **Fourier transform** of the **impulse response**,  $h(t)$ . Very important in the remainder of the course (using Laplace transforms)

This result is proven in the appendix

# Example 1: Solving a First Order ODE

Calculate the response of a CT LTI system with impulse response:

$$h(t) = e^{-bt}u(t) \quad b > 0$$

to the input signal:

$$x(t) = e^{-at}u(t) \quad a > 0$$

Taking Fourier transforms of both signals:

$$H(j\omega) = \frac{1}{b + j\omega}, \quad X(j\omega) = \frac{1}{a + j\omega}$$

gives the overall frequency response:

$$Y(j\omega) = \frac{1}{(b + j\omega)(a + j\omega)}$$

to convert this to the time domain, express as **partial fractions**:

$$Y(j\omega) = \frac{1}{b-a} \left( \frac{1}{(a + j\omega)} - \frac{1}{(b + j\omega)} \right) \quad \begin{array}{l} \text{assume} \\ b \neq a \end{array}$$

Therefore, the CT system response is:

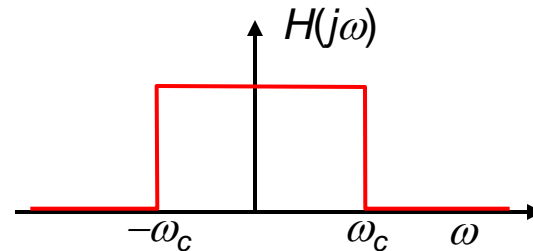
$$y(t) = \frac{1}{b-a} \left( e^{-at}u(t) - e^{-bt}u(t) \right)$$

# Example 2: Design a Low Pass Filter

Consider an ideal **low pass filter** in frequency domain:

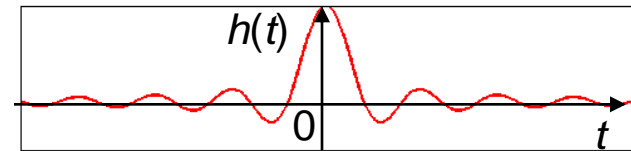
$$H(j\omega) = \begin{cases} 1 & |\omega| < \omega_c \\ 0 & |\omega| > \omega_c \end{cases}$$

$$Y(j\omega) = \begin{cases} X(j\omega) & |\omega| < \omega_c \\ 0 & |\omega| > \omega_c \end{cases}$$



The **filter's impulse response** is the **inverse Fourier transform**

$$h(t) = \frac{1}{2\pi} \int_{-\omega_c}^{\omega_c} e^{j\omega t} d\omega = \frac{\sin(\omega_c t)}{\pi t}$$



which is an ideal low pass CT filter. However it is non-causal, so this cannot be manufactured exactly & the time-domain oscillations may be undesirable

We need to approximate this filter with a causal system such as 1<sup>st</sup> order LTI system impulse response  $\{h(t), H(j\omega)\}$ :

$$a^{-1} \frac{\partial y(t)}{\partial t} + y(t) = x(t), \quad e^{-at} u(t) \stackrel{F}{\leftrightarrow} \frac{1}{a + j\omega}$$

# Lecture 11: Summary

The Fourier transform is widely used for designing **filters**. You can design systems with reject high frequency noise and just retain the low frequency components. This is natural to describe in the **frequency domain**.

Important **properties** of the Fourier transform are:

1. **Linearity and time shifts**  $ax(t) + by(t) \xleftrightarrow{F} aX(j\omega) + bY(j\omega)$

2. **Differentiation**  $\frac{dx(t)}{dt} \xleftrightarrow{F} j\omega X(j\omega)$

3. **Convolution**  $y(t) = h(t) * x(t) \xleftrightarrow{F} Y(j\omega) = H(j\omega)X(j\omega)$

Some operations are **simplified** in the frequency domain, but there are a number of signals for which the Fourier transform does not exist – this leads naturally onto **Laplace transforms**. Similar properties hold for Laplace transforms & the Laplace transform is widely used in engineering analysis.

# Lecture 11: Exercises

## Theory

1. Using linearity & time shift calculate the Fourier transform of

$$x(t) = 5e^{-3(t-1)}u(t-1) + 7e^{-3(t-2)}u(t-2)$$

2. Use the FT derivative relationship (S7) and the Fourier series/transform expression for  $\sin(\omega_0 t)$  (L10-S3) to evaluate the FT of  $\cos(\omega_0 t)$ .

3. Calculate the FTs of the systems' impulse responses

a)  $\frac{\partial y(t)}{\partial t} + 3y(t) = x(t)$

b)  $3\frac{\partial y(t)}{\partial t} + y(t) = x(t)$

4. Calculate the system responses in Q3 when the following input signal is applied

$$x(t) = e^{-5t}u(t)$$

## Matlab/Simulink

5. Verify the answer to Q1 using the Fourier transform toolbox in Matlab
6. Verify Q3 and Q4 in Simulink
7. Simulate a first order system in Simulink and input a series of sinusoidal signals with different frequencies. How does the response depend on the input frequency (S12)?

# Appendix: Proof of Convolution Property

$$y(t) = \int_{-\infty}^{\infty} x(\tau)h(t - \tau)d\tau$$

Taking Fourier transforms gives:

$$Y(j\omega) = \int_{-\infty}^{\infty} \left( \int_{-\infty}^{\infty} x(\tau)h(t - \tau)d\tau \right) e^{-j\omega t} dt$$

Interchanging the order of integration, we have

$$Y(j\omega) = \int_{-\infty}^{\infty} x(\tau) \left( \int_{-\infty}^{\infty} h(t - \tau)e^{-j\omega t} dt \right) d\tau$$

By the time shift property, the bracketed term is  $e^{-j\omega\tau}H(j\omega)$ , so

$$\begin{aligned} Y(j\omega) &= \int_{-\infty}^{\infty} x(\tau)e^{-j\omega\tau}H(j\omega)d\tau \\ &= H(j\omega) \int_{-\infty}^{\infty} x(\tau)e^{-j\omega\tau}d\tau \\ &= H(j\omega)X(j\omega) \end{aligned}$$